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Regularity for compressible isentropic Navier-Stokes equations with cylinder symmetry

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Abstract

This paper is concerned with the regularity of the global solutions in H^4 to the compressible isentropic Navier-Stokes equations with the cylinder symmetry in R^3 . Such a circular coaxial cylinder symmetric domain is an unbounded domain, but we assume that the corresponding solution depends only on one radial variable, r in $G = \{r \in R^+, 0 < a \leq r \leq b < +\infty\}$, in which the related domain G is a bounded domain. Some new ideas and more delicate estimates are introduced to prove these results.

Keywords: compressible Navier-Stokes equations; cylinder symmetry; density-dependent viscosity; regularity

1 Introduction

The compressible isentropic Navier-Stokes equations with density-dependent viscosity coefficients can be written for $t > 0$ as

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \quad (1.1)$$

$$(\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \operatorname{div}(\mu(\rho) D(\mathbf{U})) - \nabla(\lambda(\rho) \operatorname{div} \mathbf{U}) + \nabla P(\rho) = 0, \quad (1.2)$$

where $\rho(\mathbf{x}, t)$, $\mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma$ ($\gamma > 1$) stand for the fluid density, velocity, and pressure, respectively, and

$$D(\mathbf{U}) = \frac{\nabla \mathbf{U} + (\nabla \mathbf{U})^T}{2} \quad (1.3)$$

is the strain tensor and $\mu(\rho)$, $\lambda(\rho)$ are the Lamé viscosity coefficients satisfying

$$\mu(\rho) > 0, \quad \mu(\rho) + N\lambda(\rho) \geq 0. \quad (1.4)$$

In this paper we establish the regularity of global solutions to the compressible Navier-Stokes equations with cylinder symmetry in R^3 . We will pay attention to the flows between two circular coaxial cylinders. We assume that the corresponding solutions depend only on the radial variable $r \in G = \{r | 0 < a \leq r \leq b < +\infty\}$ and the time variable $t \in [0, T]$. For

simplicity, we will take $\mu(\rho) = \rho^\alpha$ and $\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho) = (\alpha - 1)\rho^\alpha$ with $\alpha > 1/2$ and $D(\mathbf{U}) = \nabla \mathbf{U}$. Then system (1.1)-(1.2) reduce to the following form:

$$\rho_t + \frac{1}{r}(r\rho u)_r = 0, \quad (1.5)$$

$$\begin{aligned} r\rho u_t + r\rho uu_r - \rho v^2 + r(\rho^\gamma)_r - (r\rho^\alpha u_r)_r \\ - (\alpha - 1)r\left(\frac{1}{r}\rho^\alpha(ru)_r\right)_r + \frac{1}{r}\rho^\alpha u = 0, \end{aligned} \quad (1.6)$$

$$r\rho v_t + r\rho uv_r + \rho uv - (r\rho^\alpha v_r)_r + \frac{1}{r}\rho^\alpha v = 0, \quad (1.7)$$

$$r\rho w_t + r\rho uw_r - (r\rho^\alpha w_r)_r = 0. \quad (1.8)$$

We consider the initial boundary value problem (1.5)-(1.8) subject to the following initial and boundary conditions:

$$(\rho, u, v, w)(r, 0) = (\rho_0, u_0, v_0, w_0)(r), \quad r \in [a, b], \quad (1.9)$$

$$u(a) = u(b) = 0, \quad v(a) = v(b) = 0, \quad w(a) = w(b) = 0. \quad (1.10)$$

First we find it convenient to transfer problems (1.5)-(1.10) into that in Lagrangian coordinates and present the desired results. It is well known that Eulerian coordinates (r, t) are connected to the Lagrangian coordinates (ξ, t) by the following relation:

$$r(\xi, t) = r_0(\xi) + \int_0^t \tilde{u}(\xi, \tau) d\tau, \quad (1.11)$$

where $\tilde{u}(\xi, t) = u(r(\xi, t), t)$ and

$$r_0(\xi) = \eta^{-1}(\xi), \quad \eta(r) = \int_a^r s\rho_0(s) ds, \quad r \in G. \quad (1.12)$$

It follows from (1.5) and the boundary condition (1.10) that

$$\frac{\partial}{\partial t} \left(\int_a^{r(\xi, t)} s\rho(s, t) ds \right) = \rho ru + \int_a^r s\rho_t ds = 0. \quad (1.13)$$

Thus,

$$\int_a^r s\rho(s, t) ds = \int_a^{r_0} s\rho_0(s) ds = \xi \quad (1.14)$$

and G is transformed to $\Omega = (0, 1)$ with

$$1 = \int_a^b s\rho(s, t) ds = \int_a^b s\rho_0(s) ds. \quad (1.15)$$

Moreover, we have

$$\partial_\xi r(\xi, t) = [r(\xi, t)\rho(r(\xi, t), t)]^{-1}. \quad (1.16)$$

For a function $\phi(r, t)$, if we write $\tilde{\phi}(\xi, t) = \phi(r(\xi, t), t)$, by virtue of (1.11), (1.16), and the chain rule, we have

$$\partial_t \tilde{\phi}(\xi, t) = \partial_t \phi(r, t) + v \partial_r \phi(r, t), \quad (1.17)$$

$$\partial_\xi \tilde{\phi}(\xi, t) = \partial_r \phi(r, t) \partial_\xi r(\xi, t) = \left(r^2 \rho(r, t)\right)^{-1} \partial_r \phi(r, t). \quad (1.18)$$

In the following, without danger of confusion we denote $(\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{w})$ still by (ρ, u, v, w) and (ξ, t) by (x, t) . Therefore, (1.5)-(1.8) in the Eulerian coordinates can be written in the Lagrangian coordinates in the new variables (x, t) as follows (see also [1]):

$$\rho_t + \rho^2 (ru)_x = 0, \quad (1.19)$$

$$\frac{u_t}{r} - \frac{v^2}{r^2} + (\rho^v)_x = \alpha (\rho^{\alpha+1} (ru)_x)_x - \frac{(\rho^\alpha)_x u}{r}, \quad (1.20)$$

$$\frac{v_t}{r} + \frac{uv}{r^2} = (\rho^{\alpha+1} (rv)_x)_x - \frac{(\rho^\alpha)_x v}{r}, \quad (1.21)$$

$$w_t - (r^2 \rho^{1+\alpha} w_x)_x = 0, \quad (1.22)$$

subject to the following initial and boundary conditions:

$$(\rho, u, v, w)(x, 0) = (\rho_0, u_0, v_0, w_0)(x), \quad x \in \Omega, \quad (1.23)$$

$$(u, v, w)(0, t) = (u, v, w)(1, t) = 0, \quad t \geq 0, \quad (1.24)$$

where $r(x, t)$ is determined by

$$\begin{aligned} r_t(x, t) &= u(x, t), & r(x, t) r_x(x, t) &= \frac{1}{\rho}, \\ r|_{t=0} &= r_0(x) = \left[a^2 + 2 \int_0^x \frac{1}{\rho_0(y, t)} dy \right]^{1/2}. \end{aligned} \quad (1.25)$$

The compressible Navier-Stokes system has been noticed academically by physicists and mathematicians for a relatively long time. We are interested in the case that the viscosity is density-dependent. Now let us first recall the related results in this direction. In the one-dimensional case, for the initial boundary value problems in a bounded domain, there have been many works (see, e.g., [2–11]) on the existence, uniqueness, and asymptotic behavior of weak solutions, based on the initial finite mass and the flow density being connected with the infinite vacuum either continuously or by a jump discontinuity. For the one-dimensional Cauchy problem, see, e.g., [12] and references therein for the results on global existence based on the density-dependent viscosity.

In two or three dimensions, the global existence and large time behavior of solution to compressible Navier-Stokes equations (1.1)-(1.2) with constant viscosity and density-dependent viscosity have been investigated for initial boundary value problems, we refer the reader to [1, 13–27] and references therein. Among them, Bresch and Desjardins [13, 14] obtained the global existence of 2D shallow water equations. For the spherically symmetric problem to a three-dimensional compressible isentropic Navier-Stokes problem, Guo *et al.* [19] analyzed the structure of the solution; Huang *et al.* [28] studied the global

well-posedness of the classical solutions with large oscillations and vacuum; Lian *et al.* [22] obtained the global existence of spherically symmetric solution for the exterior problem and the initial boundary value problem; Zhang and Fang [27] investigated the global existence and uniqueness of the weak solution without a solid core. For the cylindrically symmetric problem to the three-dimensional compressible Navier-Stokes equations, when the viscosity coefficients are both constants, the uniqueness of the weak solutions was proved in [17, 18], the global existence of isentropic compressible cylindrically symmetric solution was established in [29]; this result was later generalized to the nonisentropic case in [21]. Recently, Cui and Yao [15] proved the asymptotic behavior of a compressible p th power Newtonian fluid with cylinder symmetry; Qin [24] established the exponential stability in H^1 and H^2 for an ideal fluid. Later on, Qin and Jiang [25] proved the global existence and the exponential stability in H^4 . Jiang and Zhang [21] established a boundary layer effect and the convergence rate as the shear viscosity μ goes to zero. When the viscosity coefficient $\mu(\rho)$ is density-dependent and $\lambda(\rho)$ is a positive constant, the global existence was obtained in [26]. When viscosity coefficients μ and λ are density-dependent, Liu and Lian [1] established the global existence and asymptotic behavior of cylindrically symmetric solutions, however, there is no result on the regularity for this system.

It is noticed that the above analysis concerns the existence of solution in $H^1[0, 1]$, the regularity in $H^4[0, 1]$ has never been investigated for the three-dimensional isentropic compressible Navier-Stokes equations. Therefore, we continue the work by Liu and Lian [1] and study the regularity of the solutions in H^4 . In order to obtain a higher regularity of global strong solutions, there are many complicated estimates on higher derivations of the solution involved; this is our difficulty. To overcome this difficulty, we shall use some proper embedding theorems, and the interpolation techniques as well as many delicate estimates.

The notation in this paper will be as follows: $L^{\bar{p}}, 1 \leq \bar{p} \leq +\infty$, $W^{m, \bar{p}}, m \in \mathbb{N}$, $H^1 = W^{1, 2}$, $H_0^1 = W_0^{1, 2}$ denote the usual (Sobolev) spaces on $[0, 1]$. In addition, $\|\cdot\|_B$ denotes the norm in the space B ; we also put $\|\cdot\| = \|\cdot\|_{L^2}$. Subscripts t and x denote the (partial) derivatives with respect to t and x , respectively. We use C_i ($i = 1, 2, 4$) to denote the generic positive constant depending only on the H^i norm of the initial data (ρ_0, u_0, v_0, w_0) and the variable t .

Before stating the main result, we assume the initial data

$$(\rho_0 - \bar{\rho}, u_0, v_0, w_0) \in (H^4[0, 1])^4, \quad (1.26)$$

with $\rho_0 > 0$ and $\bar{\rho} = \frac{1}{b-a} \int_a^b \rho_0 r dr$, and we define

$$\begin{aligned} H_0 &= \int_0^1 \left(\frac{1}{2} (u_0^2 + v_0^2 + w_0^2) + \frac{1}{\gamma-1} (\rho_0^{\gamma-1} - \bar{\rho}^{\gamma-1}) + \bar{\rho}^\gamma \left(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}} \right) \right) dx, \\ H_1 &= \int_0^1 \left(\frac{1}{2} ((u_0 + r(\rho_0^\alpha)_x)^2 + v_0^2 + w_0^2) + \frac{1}{\gamma-1} (\rho_0^{\gamma-1} - \bar{\rho}^{\gamma-1}) + \bar{\rho}^\gamma \left(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}} \right) \right) dx. \end{aligned}$$

Now we are in a position to state our main results.

Theorem 1.1 *Let $\gamma > 1$, $\alpha > 1/2$. Assume that the initial data satisfies (1.26) and $H_0(H_0 + H_1) < a^2 \alpha^2 (2\alpha - 1)^{-2} \bar{\rho}^{\gamma+2\alpha-1}$. Then there exists a unique generalized global solution*

$(\rho(t), u(t), v(t), w(t)) \in (H^4[0, 1])^4$ to the problem (1.19)-(1.24) verifying that, for $T > 0$,

$$\rho - \bar{\rho} \in L^\infty([0, T], H^4[0, 1]) \cap L^2([0, T], H^4[0, 1]), \quad (1.27)$$

$$(u, v, w) \in L^\infty([0, T], H^4[0, 1]) \cap L^2([0, T], H^5[0, 1]), \quad (1.28)$$

$$(u_t, v_t, w_t) \in L^\infty([0, T], H^2[0, 1]) \cap L^2([0, T], H^3[0, 1]), \quad (1.29)$$

$$(u_{tt}, v_{tt}, w_{tt}) \in L^\infty([0, T], L^2[0, 1]) \cap L^2([0, T], H^1[0, 1]). \quad (1.30)$$

Corollary 1.1 *Under the assumptions of Theorem 1.1, (1.27)-(1.28) implies $(\rho(t), u(t), v(t), w(t))$ is the classical solution verifying, for any $t > 0$,*

$$\|\rho(t) - \bar{\rho}\|_{C^{3+1/2}} + \|u(t)\|_{C^{3+1/2}} + \|v(t)\|_{C^{3+1/2}} + \|w(t)\|_{C^{3+1/2}} \leq C_4. \quad (1.31)$$

The rest of the paper is arranged as follows. Section 2 is concerned with the proof of the regularity for a cylindrically symmetric solution to the compressible Navier-Stokes problem in detail.

2 Proof of Theorem 1.1

We will complete the proof of Theorem 1.1 and assume that the assumptions in Theorem 1.1 are valid. We begin with the following lemma.

Lemma 2.1 *Under the assumptions in Theorem 1.1, there exist positive constants $\rho_* > 0$ and $\rho^* > 0$ with $\rho_* < \bar{\rho} < \rho^*$ so that the unique global solution $(\rho(t), u(t), v(t), w(t))$ to problem (1.19)-(1.24) exists and satisfies, for any $T > 0$,*

$$0 < \rho_* \leq \rho(x, t) \leq \rho^*, \quad (2.1)$$

$$(\rho - \bar{\rho})_x \in L^\infty([0, T], L^2[0, 1]) \cap L^2([0, T], L^2[0, 1]), \quad (2.2)$$

$$(u, v, w) \in L^\infty([0, T], H^2[0, 1]) \cap L^2([0, T], H^3[0, 1]), \quad (2.3)$$

$$\|(u_t, v_t, w_t)(t)\|^2 + \int_0^t \|(u_t, v_t, w_t)(s)\|_{H^1}^2 ds \leq C_2, \quad \forall t \in [0, T], \quad (2.4)$$

where $\bar{\rho}$ is the same as in Theorem 1.1.

Proof Estimates (2.1)-(2.4) were obtained in Ref. [1], the proof is complete. \square

Lemma 2.2 *Under the assumptions in Theorem 1.1, the following estimate holds for any $T > 0$:*

$$\|\rho_{xx}(t)\|^2 + \int_0^t \|\rho_{xx}(s)\|^2 ds \leq C_2, \quad t \in [0, T]. \quad (2.5)$$

Proof Differentiating (1.20) with respect to x , exploiting (1.19), we have

$$\frac{u_{tx}}{r} = (-\rho^\gamma + \alpha\rho^{\alpha+1}(ru)_x)_{xx} + \left(\frac{v^2}{r^2} - \frac{(\rho^\alpha)_x u}{r}\right)_x + \frac{u_t}{\rho r^3}, \quad (2.6)$$

which gives

$$\alpha(\rho^{\alpha-1}\rho_{xx})_t + \gamma\rho^{\gamma-1}\rho_{xx} = M_0(x, t) \quad (2.7)$$

with

$$\begin{aligned} M_0(x, t) = & \frac{2v v_x}{r^2} - \frac{2v^2 r_x}{r^3} - \alpha(\alpha-1)(2\rho^{\alpha-2}\rho_x\rho_{tx} - (\alpha-2)\rho^{\alpha-3}\rho_t\rho_x^2) \\ & - \gamma(\gamma-1)\rho^{\gamma-2}\rho_x^2 - \frac{(\rho^\alpha)_{xx}u + (\rho^\alpha)_x u}{r} - \frac{(\rho^\alpha)_x u}{\rho r^3} - \frac{u_{tx}}{r} + \frac{u_t}{\rho r^3}. \end{aligned}$$

Multiplying (2.7) by $\rho^{\alpha-1}\rho_{xx}$, integrating the result over $[0, 1]$, we deduce

$$\begin{aligned} & \frac{d}{dt} \|\rho^{\alpha-1}\rho_{xx}(t)\|^2 + \int_0^1 \gamma\rho^{\gamma+\alpha-2}\rho_{xx}^2 dx \\ & \leq C_1 \int_0^1 \|\rho_{xx}\|(\|v_x\| + \|v^2\| + \|\rho_x\rho_{tx}\| + \|\rho_t\rho_x^2\| + \|\rho_x^2\| \\ & \quad + \|\rho_x\| + \|u_{tx}\| + \|u_t\|) - \int_0^1 \alpha \frac{u}{r} \rho^{2\alpha-2}\rho_{xx}^2 dx, \end{aligned}$$

which, by Young's inequality and the interpolation inequality, implies

$$\frac{d}{dt} \|\rho^{\alpha-1}\rho_{xx}(t)\|^2 \leq C_1(\|\rho_{xx}\|^2 + \|\rho_x\|^2 + \|u_x\|_{H^1}^2 + \|u_t\|_{H^1}^2). \quad (2.8)$$

Integrating (2.8) with respect to t over $[0, T]$, using initial condition (1.26) and Lemma 2.1, we derive

$$\|\rho_{xx}(t)\|^2 \leq C_2 + C_1 \int_0^t \|\rho_{xx}(s)\|^2 ds, \quad \forall t \in [0, T],$$

which, by virtue of Gronwall's inequality, gives (2.5). The proof is complete. \square

Lemma 2.3 *Under the assumptions in Theorem 1.1, the following estimates hold for any $T > 0$:*

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{tx}(s)\|^2 ds \leq C_4 + C_2 \int_0^t (\|u_{txx}\|^2 + \|v_{txx}\|^2)(s) ds, \quad t \in [0, T], \quad (2.9)$$

$$\|v_{tt}(t)\|^2 + \int_0^t \|v_{tx}(s)\|^2 ds \leq C_4 + C_2 \int_0^t (\|v_{txx}\|^2 + \|u_{txx}\|^2)(s) ds, \quad t \in [0, T], \quad (2.10)$$

$$\|w_{tt}(t)\|^2 + \int_0^t \|w_{tx}(s)\|^2 ds \leq C_4, \quad t \in [0, T]. \quad (2.11)$$

Proof We infer from (1.20)-(1.22) and Lemmas 2.1-2.2 that

$$\|u_t(t)\| \leq C_1(\|\rho_x(t)\| + \|u_x(t)\|_{H^1} + \|v_x(t)\|), \quad (2.12)$$

$$\|v_t(t)\| \leq C_1(\|\rho_x(t)\| + \|u_x(t)\| + \|v_x(t)\|_{H^1}), \quad (2.13)$$

$$\|w_t(t)\| \leq C_1(\|w_x(t)\|_{H^1} + \|\rho_x(t)\|). \quad (2.14)$$

Differentiating (1.20)-(1.22) with respect to x , respectively, and exploiting Lemmas 2.1-2.2, we have

$$\|u_{tx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^1} + \|v_x(t)\| + \|u_x(t)\|_{H^2}), \quad (2.15)$$

$$\|v_{tx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^1} + \|u_x(t)\| + \|v_x(t)\|_{H^2}), \quad (2.16)$$

$$\|w_{tx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^1} + \|w_x(t)\|_{H^2}), \quad (2.17)$$

or

$$\|u_{xxx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^1} + \|v_x(t)\| + \|u_x(t)\|_{H^1} + \|u_{tx}(t)\|), \quad (2.18)$$

$$\|v_{xxx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^1} + \|u_x(t)\| + \|v_x(t)\|_{H^1} + \|v_{tx}(t)\|), \quad (2.19)$$

$$\|w_{xxx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^1} + \|w_x(t)\|_{H^1} + \|w_{tx}(t)\|). \quad (2.20)$$

Differentiating (1.20)-(1.22) with respect to x twice, respectively, using Lemmas 2.1-2.2, we get

$$\|u_{txx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^2} + \|u_t(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|u_x(t)\|_{H^3}), \quad (2.21)$$

$$\|v_{txx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^2} + \|v_t(t)\|_{H^1} + \|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^3}), \quad (2.22)$$

$$\|w_{txx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^2} + \|w_x(t)\|_{H^3}), \quad (2.23)$$

or

$$\|u_{xxxx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^2} + \|u_t(t)\|_{H^2} + \|v_x(t)\|_{H^1} + \|u_x(t)\|_{H^2}), \quad (2.24)$$

$$\|v_{xxxx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^2} + \|v_t(t)\|_{H^2} + \|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2}), \quad (2.25)$$

$$\|w_{xxxx}(t)\| \leq C_1(\|\rho_x(t)\|_{H^2} + \|w_x(t)\|_{H^2} + \|w_{txx}(t)\|). \quad (2.26)$$

Differentiating (1.20)-(1.22) with respect to t , respectively, we deduce

$$\|u_{tt}(t)\| \leq C_1(\|u_t(t)\|_{H^2} + \|v_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}), \quad (2.27)$$

$$\|v_{tt}(t)\| \leq C_1(\|v_t(t)\|_{H^2} + \|u_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}), \quad (2.28)$$

$$\|w_{tt}(t)\| \leq C_1(\|u_{tx}(t)\|_{H^1} + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}). \quad (2.29)$$

Now differentiating (1.20) with respect to t twice, multiplying the resulting equation by $(\frac{u}{r})_{tt}$ in $L^2[0, 1]$, and using integration by parts and (1.19), we conclude

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{u}{r} \right)_{tt}^2 dx &= - \int_0^1 (-\rho^\gamma + \alpha \rho^{\alpha+1} (\rho u)_x)_{tt} \left(\frac{u}{r} \right)_{tt} dx \\ &\quad + \int_0^1 \left(\frac{v^2 - u^2}{r^2} \right)_{tt} \left(\frac{u}{r} \right)_{tt} dx - \int_0^1 \left(\frac{(\rho^\alpha)_x u}{r} \right)_{tt} \left(\frac{u}{r} \right)_{tt} dx \\ &= A_1 + A_2 + A_3. \end{aligned} \quad (2.30)$$

Employing Lemmas 2.1-2.2, (2.27)-(2.28), (1.19), and the interpolation inequality, we get, for any small $\varepsilon \in (0, 1)$,

$$\begin{aligned} A_1 &= - \int_0^1 \left(-\rho^\gamma + \alpha \rho^{\alpha+1} (\rho u)_x \right)_{tt} \left(\frac{u}{r} \right)_{tx} dx \\ &\leq -\alpha \int_0^1 \frac{\rho^{\alpha+2}}{r} u_{tx}^2 dx + \varepsilon \|u_{tx}(t)\|^2 + C_2 (\|\rho_{tt}(t)\|_{H^1}^2 + \|\rho_t(t)\|_{H^1}^2 \\ &\quad + \|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 + \|u_x(t)\|_{H^1}^2 + \|\rho_x(t)\|_{H^1}^2) \\ &\leq -C_2^{-1} \|u_{tx}(t)\|^2 + C_2 (\|u_x(t)\|_{H^2}^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_t(t)\|_{H^2}^2), \end{aligned} \quad (2.31)$$

$$\begin{aligned} A_2 &= \int_0^1 \left(\frac{v^2 - u^2}{r^2} \right)_{tt} \left(\frac{u}{r} \right)_{tt} dx \\ &\leq C_2 (\|u_t(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 + \|u_x(t)\|^2) \\ &\leq C_2 (\|u_t(t)\|_{H^2}^2 + \|v_t(t)\|_{H^2}^2 + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}), \end{aligned} \quad (2.32)$$

$$\begin{aligned} A_3 &= - \int_0^1 \left(\frac{(\rho^\alpha)_x u}{r} \right)_{tt} \left(\frac{u}{r} \right)_{tt} dx \\ &\leq C_2 (\|u_{tt}(t)\|^2 + \|u_t(t)\|_{H^1}^2 + \|\rho_{tx}(t)\|^2 + \|\rho_{ttx}(t)\|^2 + \|\rho_x(t)\|^2) \\ &\leq C_2 (\|u_t(t)\|_{H^2}^2 + \|\rho_x(t)\|^2 + \|u_x(t)\|_{H^1}^2). \end{aligned} \quad (2.33)$$

Integrating (2.30) with respect to t , applying Lemmas 2.1-2.2, initial condition (1.26), and (2.31)-(2.33), we obtain (2.9).

Differentiating (1.21) with respect to t twice, multiplying the resulting equation by $(\frac{v}{r})_{tt}$ in $L^2[0, 1]$, and using integration by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{v}{r} \right)_{tt}^2 dx &= - \int_0^1 (\rho^{\alpha+1} (\rho v)_x)_{tt} \left(\frac{v}{r} \right)_{tx} dx - \int_0^1 \left(\frac{(\rho^\alpha)_x v}{r} \right)_{tt} \left(\frac{v}{r} \right)_{tt} dx \\ &= B_1 + B_2, \end{aligned} \quad (2.34)$$

where

$$B_1(t) = - \int_0^1 (\rho^{\alpha+1} (\rho v)_x)_{tt} \left(\frac{v}{r} \right)_{tx} dx, \quad B_2(t) = - \int_0^1 \left(\frac{(\rho^\alpha)_x v}{r} - \frac{2uv}{r^2} \right)_{tt} \left(\frac{v}{r} \right)_{tt} dx.$$

Using the interpolation inequality, Lemmas 2.1-2.2 and (2.27)-(2.28), we obtain for $\varepsilon \in (0, 1)$,

$$\begin{aligned} B_1(t) &\leq - \int_0^1 \frac{\rho^{\alpha+2}}{r} v_{tx}^2 dx + \varepsilon \|v_{tx}(t)\|^2 + C_2 (\|u_t(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 + \|v_{tt}(t)\|^2 \\ &\quad + \|\rho_{tx}(t)\|^2 + \|\rho_{ttx}(t)\|^2 + \|\rho_x(t)\|^2 + \|v_x(t)\|^2) \\ &\leq - \int_0^1 \frac{\rho^{\alpha+2}}{r} v_{tx}^2 dx + \varepsilon \|v_{tx}(t)\|^2 + C_2 (\|v_x(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 \\ &\quad + \|u_t(t)\|_{H^2}^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_x(t)\|_{H^1}^2), \end{aligned} \quad (2.35)$$

$$\begin{aligned}
B_2(t) &\leq C_2(\|v_{tt}(t)\|^2 + \|v_t(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 + \|\rho_{tx}(t)\|^2 \\
&\quad + \|\rho_{tx}(t)\|^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_x(t)\|_{H^1}^2) \\
&\leq C_2(\|v_t(t)\|_{H^2}^2 + \|u_t(t)\|_{H^2}^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_x(t)\|_{H^1}^2).
\end{aligned} \quad (2.36)$$

Integrating (2.34) with respect to t , using Lemmas 2.1-2.2 and (2.35)-(2.36), and picking ϵ small enough, we conclude (2.10).

Similarly, differentiating (1.22) with respect to t twice, multiplying the resulting equation by w_{tt} in $L^2[0, 1]$, and using integration by parts, we deduce

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^1 w_{tt}^2(x, t) dx \\
&= - \int_0^1 (r^2 \rho^{1+\alpha} w_x)_{tt} w_{ttx} dx \\
&\leq - \int_0^1 r^2 \rho^{1+\alpha} w_{ttx}^2(x, t) dx + C_2(\|w_x\| + \|u_t w_x\| + \|\rho_t w_x\| + \|\rho_t w_{tx}\| \\
&\quad + \|\rho_{tt} w_x\| + \|\rho_t w_{tx}\|) \|w_{ttx}\| \\
&\leq -C_1^{-1} \|w_{ttx}\|^2 + C_2(\|w_x(t)\|^2 + \|u_t(t)\|_{H^1}^2 + \|w_t(t)\|_{H^1}^2 + \|u_x(t)\|^2).
\end{aligned} \quad (2.37)$$

Integrating (2.37) with respect to t , using Lemmas 2.1-2.2, we derive the estimate (2.11). The proof is complete. \square

Lemma 2.4 *Under the assumptions in Theorem 1.1, the following estimates hold for any $T > 0$ and $\epsilon \in (0, 1)$:*

$$\|u_{tx}(t)\|^2 + \int_0^t \|u_{txx}(s)\|^2 ds \leq C_4 + C_2^{-1} \epsilon^2 \int_0^t \|u_{ttx}(s)\|^2 ds, \quad t \in [0, T], \quad (2.38)$$

$$\|v_{tx}(t)\|^2 + \int_0^t \|v_{txx}(s)\|^2 ds \leq C_4 + C_2^{-1} \epsilon^2 \int_0^t \|v_{ttx}(s)\|^2 ds, \quad t \in [0, T], \quad (2.39)$$

$$\|w_{tx}(t)\|^2 + \int_0^t \|w_{txx}(s)\|^2 ds \leq C_4, \quad t \in [0, T]. \quad (2.40)$$

Proof Differentiating (1.20) with respect to t and x , then multiplying the result by $(\frac{u}{r})_{tx}$ in $L^2[0, 1]$, and integrating by parts, we deduce that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{u}{r}\right)_{tx}^2 dx \\
&= (-\rho^\gamma + \alpha \rho^{\alpha+1} (\rho u)_x)_{tx} \left(\frac{u}{r}\right)_{tx} \Big|_0^1 - \int_0^1 (-\rho^\gamma + \alpha \rho^{\alpha+1} (\rho u)_x)_{tx} \left(\frac{u}{r}\right)_{txx} dx \\
&\quad + \int_0^1 \left(\frac{v^2 - u^2}{r^2} - \frac{(\rho^\alpha)_x u}{r}\right)_{tx} \left(\frac{u}{r}\right)_{tx} dx \\
&= D_0(x, t) + D_1(t) + D_2(t),
\end{aligned} \quad (2.41)$$

where

$$\begin{aligned} D_0(x, t) &= \left(-\rho^\gamma + \alpha \rho^{\alpha+1} (\rho u)_x \right)_{tx} \left(\frac{u}{r} \right)_{tx} \Big|_0^1, \\ D_1(t) &= - \int_0^1 \left(-\rho^\gamma + \alpha \rho^{\alpha+1} (\rho u)_x \right)_{tx} \left(\frac{u}{r} \right)_{txx} dx, \\ D_2(t) &= \int_0^1 \left(\frac{v^2 - u^2}{r^2} - \frac{(\rho^\alpha)_x u}{r} \right)_{tx} \left(\frac{u}{r} \right)_{tx} dx. \end{aligned}$$

Now employing Lemmas 2.1-2.2 and the interpolation inequality, using Young's inequality several times, we have, for $\varepsilon \in (0, 1)$,

$$\begin{aligned} D_0(x, t) &\leq C_2 \left(\|\rho_x\|_{L^\infty}^2 + \|\rho_x u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|\rho_x^2 u_x\|_{L^\infty} \right. \\ &\quad + \|\rho_x u_x^2\|_{L^\infty} + \|u_x u_{xx}\|_{L^\infty} + \|\rho_x u_{xx}\|_{L^\infty} + \|\rho_x u_{tx}\|_{L^\infty} \\ &\quad \left. + \|u_x^2\|_{L^\infty} + \|u_{txx}\|_{L^\infty} \right) \|u_{tx}\|_{L^\infty} \\ &\leq C_2 \left(\|u_x\|_{H^2} + \|\rho_x\|_{H^1} + \|u_{tx}\|^{\frac{1}{2}} \|u_{txx}\|^{\frac{1}{2}} \right. \\ &\quad \left. + \|u_{txx}\|^{\frac{1}{2}} \|u_{txxx}\|^{\frac{1}{2}} \right) \|u_{tx}\|^{\frac{1}{2}} \|u_{txx}\|^{\frac{1}{2}} \\ &\leq C_2^{-1} \varepsilon^2 \left(\|u_{txx}\|^2 + \|u_{txxx}\|^2 \right) \\ &\quad + C_2 \varepsilon^{-6} \left(\|u_{tx}\|^2 + \|u_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2 \right), \end{aligned} \quad (2.42)$$

$$\begin{aligned} D_1(t) &\leq -\alpha \int_0^1 \rho^{\alpha+1} u_{txx}^2 dx + \varepsilon \|u_{txx}(t)\|^2 \\ &\quad + C_2 \left(\|\rho_x(t)\|_{H^1}^2 + \|u_x(t)\|_{H^2}^2 \right. \\ &\quad \left. + \|u_{tx}(t)\|^2 + \|u_t(t)\|^2 \right), \end{aligned} \quad (2.43)$$

$$\begin{aligned} D_2(t) &\leq C_2 \left(\|u_x u_t\| + \|u_t\|_{H^1} + \|v_t\|_{H^1} + \|u_x\| + \|v_x\| + \|u^3\| + \|v^2\| \right. \\ &\quad + \|\rho_{tx}\|_{H^1} + \|\rho_{tx} u_x\| + \|\rho_{xx} u_t\| + \|\rho_x u_{tx}\| + \|\rho_{xx}\| \\ &\quad \left. + \|\rho_x u_x\| \right) \left(\|u_{tx}\| + \|u_t\| + \|u_x\| + \|u^2\| \right) \\ &\leq C_2 \left(\|u_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2 + \|v_t\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|v_x\|_{H^1}^2 \right). \end{aligned} \quad (2.44)$$

On the other hand, we differentiate (1.20)-(1.22) with respect to x and t , and use Lemmas 2.1-2.2 and (2.12)-(2.29) to conclude

$$\|u_{txxx}(t)\| \leq C_2 \left(\|u_x(t)\|_{H^2} + \|\rho_x(t)\|_{H^1} + \|u_t(t)\|_{H^2} + \|u_{txx}(t)\| \right), \quad (2.45)$$

$$\|v_{txxx}(t)\| \leq C_2 \left(\|u_x(t)\|_{H^2} + \|\rho_x(t)\|_{H^1} + \|v_t(t)\|_{H^2} + \|v_{txx}(t)\| \right), \quad (2.46)$$

$$\|w_{txxx}(t)\| \leq C_2 \left(\|\rho_x(t)\|_{H^1} + \|w_{tx}(t)\|_{H^1} + \|w_{txx}(t)\| \right). \quad (2.47)$$

Integrating (2.41) with respect to t , using (2.42)-(2.45) and Lemmas 2.1-2.2, we obtain (2.38).

Analogously, differentiating (1.21) with respect to t and x , then multiplying the resultant by $(\frac{v}{r})_{tx}$ in $L^2[0, 1]$, and integrating by parts, we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{v}{r} \right)_{tx}^2 dx \\ &= (\rho^{\alpha+1}(\rho v)_x)_{tx} \left(\frac{v}{r} \right)_{tx} \Big|_0^1 - \int_0^1 (\rho^{\alpha+1}(\rho v)_x)_{tx} \left(\frac{v}{r} \right)_{txx} dx \\ & \quad + \int_0^1 \left(\frac{2uv}{r^2} - \frac{(\rho^\alpha)_x v}{r} \right)_{tx} \left(\frac{v}{r} \right)_{tx} dx \\ &= E_0(x, t) + E_1(t) + E_2(t), \end{aligned} \quad (2.48)$$

where

$$\begin{aligned} E_0(x, t) &= (\rho^{\alpha+1}(\rho v)_x)_{tx} \left(\frac{v}{r} \right)_{tx} \Big|_0^1, & E_1(t) &= - \int_0^1 (\rho^{\alpha+1}(\rho v)_x)_{tx} \left(\frac{v}{r} \right)_{txx} dx, \\ E_2(t) &= \int_0^1 \left(\frac{2uv}{r^2} - \frac{(\rho^\alpha)_x v}{r} \right)_{tx} \left(\frac{v}{r} \right)_{tx} dx. \end{aligned}$$

Now we apply the interpolation inequality and Young's inequality to estimate $E_0(x, t)$, $E_1(t)$, $E_2(t)$ for any $\varepsilon > 0$,

$$\begin{aligned} E_0(x, t) &\leq C_2 (\|v_{txx}\|_{L^\infty} + \|v_{tx}\|_{L^\infty} + \|v_t \rho_x\|_{L^\infty} + \|v_t\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|u_x v_x\|_{L^\infty} \\ & \quad + \|v_{xx}\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty}) \|v_{tx}\|_{L^\infty} \\ &\leq C_2 (\|u_x\|_{H^2} + \|v_x\|_{H^2} + \|\rho_x\|_{H^1} + \|v_t\|_{H^1} \\ & \quad + \|v_{tx}\|^{\frac{1}{2}} \|v_{txx}\|^{\frac{1}{2}} + \|v_{txx}\|^{\frac{1}{2}} \|v_{txxx}\|^{\frac{1}{2}}) \|v_{tx}\|^{\frac{1}{2}} \|v_{txx}\|^{\frac{1}{2}} \\ &\leq C_2^{-1} \varepsilon^2 (\|v_{txx}(t)\|^2 + \|v_{txxx}(t)\|^2) + C_2 \varepsilon^{-6} (\|v_{tx}(t)\|^2 + \|u_x(t)\|_{H^2}^2 \\ & \quad + \|v_x(t)\|_{H^2} + \|\rho_x(t)\|_{H^1}), \end{aligned} \quad (2.49)$$

$$\begin{aligned} E_1(t) &\leq - \int_0^1 \rho^{\alpha+1} v_{txx}^2 dx + \frac{1}{2} \int_0^1 \rho^{\alpha+1} v_{txx}^2 dx + C_2 (\|u_x\|_{H^2}^2 + \|v_x\|_{H^1}^2 \\ & \quad + \|\rho_x\|_{H^1}^2 + \|v_t\|^2 + \|v_{tx}\|^2), \end{aligned} \quad (2.50)$$

$$\begin{aligned} E_2(t) &\leq C_2 (\|u_{tx}\| + \|u_t v_x\| + \|u_x v_t\| + \|v_{tx}\| + \|u_t\| + \|v_t\| + \|u_x\| \\ & \quad + \|v_x\| + \|\rho_{txx}\| + \|\rho_{xt} v_x\| + \|\rho_{xx} v_t\| + \|\rho_x v_{tx}\| + \|\rho_{xt}\| + \|\rho_x v_t\| \\ & \quad + \|\rho_{xx}\| + \|\rho_x v_x\| + \|\rho_x u_x\|) (\|v_{tx}\| + \|v_t\| + \|u_x\| + \|v_x\|) \\ &\leq C_2 (\|u_t\|_{H^1}^2 + \|v_t\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|\rho_x\|_{H^1}^2 + \|v_x\|_{H^1}^2). \end{aligned} \quad (2.51)$$

Inserting (2.46) into (2.49), integrating (2.48) with respect to t , and using (2.49)-(2.51), we can get (2.39).

Differentiating (1.22) with respect to x and t , multiplying the resulting equation by w_{tx} in $L^2[0, 1]$, integrating by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w_{tx}\|^2 &= (r^2 \rho^{1+\alpha} w_x)_{tx} w_{tx} \Big|_0^1 - \int_0^1 (r^2 \rho^{1+\alpha} w_x)_{tx} w_{txx} dx \\ &= F_0(x, t) + F_1(t). \end{aligned} \quad (2.52)$$

Employing Lemmas 2.1-2.2, the interpolation inequality and (2.47), we infer for any $\varepsilon \in (0, 1)$ that

$$\begin{aligned} F_0(x, t) &= (r^2 \rho^{1+\alpha} w_x)_{tx} w_{tx} \Big|_0^1 \\ &\leq C_2 (\|w_x\|_{L^\infty} + \|w_x u_x\|_{L^\infty} + \|\rho_x w_x\|_{L^\infty} + \|w_{xx}\|_{L^\infty} \\ &\quad + \|w_{tx}\|_{L^\infty} + \|w_{txx}\|_{L^\infty}) \|w_{tx}\|_{L^\infty} \\ &\leq \varepsilon^2 (\|w_{txx}\|^2 + \|w_{txxx}\|^2) + C_2 (\|w_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|w_{tx}\|^2) \\ &\leq \varepsilon^2 (\|w_{txx}\|^2 + \|w_{txx}\|^2) + C_2 (\|w_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|w_{tx}\|^2), \end{aligned} \quad (2.53)$$

$$\begin{aligned} F_1(t) &= - \int_0^1 (r^2 \rho^{1+\alpha} w_x)_{tx} w_{txx} dx \\ &\leq - \int_0^1 r^2 \rho^{1+\alpha} w_{txx}^2 dx + C_2 (\|w_x\| + \|w_x u_x\| + \|w_x \rho_x\| + \|w_{xx}\| \\ &\quad + \|w_{xt}\| + \|\rho_x w_{tx}\|) \|w_{txx}\| \\ &\leq -C_1^{-1} \|w_{txx}\|^2 + C_2 (\|w_x\|_{H^1}^2 + \|w_t\|_{H^1}^2 + \|\rho_x\|_{H^1}^2). \end{aligned} \quad (2.54)$$

Integrating (2.52) with respect to t , picking ε small enough, using Lemmas 2.1-2.3 and (2.53)-(2.54), we derive the estimate (2.40). The proof is complete. \square

Lemma 2.5 *Under the assumptions in Theorem 1.1, the following estimates hold for any $T > 0$:*

$$\|u_{tt}(t)\|^2 + \|u_{tx}(t)\|^2 + \int_0^t (\|u_{txx}\|^2 + \|u_{txx}\|^2)(s) ds \leq C_4, \quad t \in [0, T], \quad (2.55)$$

$$\|v_{tt}(t)\|^2 + \|v_{tx}(t)\|^2 + \int_0^t (\|v_{txx}\|^2 + \|v_{txx}\|^2)(s) ds \leq C_4, \quad t \in [0, T], \quad (2.56)$$

$$\begin{aligned} &\|\rho_{xxx}(t)\|^2 + \|u_{xxx}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|w_{xxx}(t)\|^2 + \int_0^t (\|\rho_{xxx}\|^2 + \|u_{xxx}\|^2 \\ &\quad + \|v_{xxx}\|^2 + \|w_{xxx}\|^2)(s) ds \leq C_4, \quad t \in [0, T]. \end{aligned} \quad (2.57)$$

Proof Adding (2.38) and (2.39), using (2.9)-(2.10) and picking ε small enough, we easily obtain (2.55)-(2.56).

Differentiating (2.7) with respect to x , we have

$$\alpha (\rho^{\alpha-1} \rho_{xxx})_t + \gamma \rho^{\gamma-1} \rho_{xxx} = M_1(x, t), \quad (2.58)$$

where

$$M_1(x, t) = M_{0x}(x, t) - (\alpha - 1)(\rho^{\alpha-2} \rho_x \rho_{xx})_t - \gamma(\gamma - 1) \rho_x \rho_{xx}.$$

An easy calculation with the interpolation inequality, Lemmas 2.1-2.2, and (2.55)-(2.56) gives

$$\begin{aligned} \|M_1(t)\| &\leq C_2(\|M_{0x}\| + \|\rho_x \rho_{xx}\| + \|\rho_{xt} \rho_{xx}\| + \|\rho_x \rho_{txx}\|) \\ &\leq C_2(\|v_x^2\| + \|v_{xx}\| + \|\rho_x \rho_{txx}\| + \|\rho_{xx} \rho_{tx}\| + \|\rho_x^2 \rho_{tx}\| + \|\rho_t \rho_x \rho_{xx}\| \\ &\quad + \|\rho_x \rho_{xx}\| + \|\rho_{xxx}\| + \|\rho_{xx} u_x\| + \|u_{tx}\| + \|u_{txx}\| + \|u_t\| + \|u_t \rho_x\|) \\ &\leq C_2(\|\rho_x(t)\|_{H^2}^2 + \|u_x(t)\|_{H^2}^2 + \|v_x(t)\|_{H^1}^2 + \|u_t(t)\|_{H^2}^2), \end{aligned}$$

which, along with Lemmas 2.1-2.2 and (2.55)-(2.56), implies

$$\int_0^t \|M_1(s)\|^2 ds \leq C_4 + C_2 \int_0^t \|\rho_{xxx}(s)\|^2 ds, \quad \forall t \in [0, T]. \quad (2.59)$$

Multiplying (2.58) by $\rho^{\alpha-1} \rho_{xxx}$ in $L^2[0, 1]$, we deduce

$$\frac{\alpha}{2} \frac{d}{dt} \|\rho^{\alpha-1} \rho_{xxx}\|^2 + \gamma \int_0^1 \rho^{\gamma+\alpha-2} \rho_{xxx}^2 dx \leq C_1 \|M_1(t)\|^2, \quad (2.60)$$

which implies

$$\frac{d}{dt} \|\rho^{\alpha-1} \rho_{xxx}\|^2 \leq C_1 \|M_1(t)\|^2. \quad (2.61)$$

Integrating (2.61) with respect to t , using (2.59), we conclude

$$\|\rho^{\alpha-1} \rho_{xxx}\|^2 \leq C_4 + C_2 \int_0^t \|\rho_{xxx}(s)\|^2 ds,$$

which, by virtue of Gronwall's inequality and (2.60), gives

$$\|\rho_{xxx}(t)\|^2 + \int_0^t \|\rho_{xxx}(s)\|^2 ds \leq C_4, \quad \forall t \in [0, T]. \quad (2.62)$$

By (2.18)-(2.20), (2.55)-(2.56), and Lemmas 2.1-2.4, we conclude

$$\|u_{xxx}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|w_{xxx}(t)\|^2 \leq C_4, \quad \forall t \in [0, T]. \quad (2.63)$$

By virtue of (2.24)-(2.26), (2.55)-(2.56), (2.62), and Lemmas 2.1-2.4, we can get

$$\int_0^t (\|u_{xxxx}\|^2 + \|v_{xxxx}\|^2 + \|w_{xxxx}\|^2)(s) ds \leq C_4, \quad t \in [0, T],$$

which, along with (2.62)-(2.63), gives (2.57). The proof is complete. \square

Lemma 2.6 *Under the assumptions in Theorem 1.1, the following estimates hold for any $T > 0$:*

$$\|u_{txx}(t)\|^2 + \|v_{txx}(t)\|^2 + \|w_{txx}(t)\|^2 \leq C_4, \quad t \in [0, T], \quad (2.64)$$

$$\|\rho_{xxxx}(t)\|^2 + \int_0^t \|\rho_{xxxx}(s)\|^2 ds \leq C_4, \quad t \in [0, T], \quad (2.65)$$

$$\begin{aligned} &\|u_{xxxx}(t)\|^2 + \|v_{xxxx}(t)\|^2 + \|w_{xxxx}(t)\|^2 + \int_0^t (\|u_{xxxx}\|^2 \\ &\quad + \|v_{xxxx}\|^2 + \|w_{xxxx}\|^2)(s) ds \leq C_4, \quad t \in [0, T]. \end{aligned} \quad (2.66)$$

Proof Differentiating (1.20)-(1.22) with respect to t , respectively, we deduce

$$\|u_{txx}(t)\| \leq C_1(\|u_t(t)\|_{H^1} + \|v_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1} + \|u_{tt}(t)\|), \quad (2.67)$$

$$\|v_{txx}(t)\| \leq C_1(\|v_t(t)\|_{H^1} + \|u_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1} + \|v_{tt}(t)\|), \quad (2.68)$$

$$\|w_{txx}(t)\| \leq C_1(\|w_{tx}(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1} + \|w_x(t)\|_{H^1} + \|w_{tt}(t)\|). \quad (2.69)$$

By virtue of Lemmas 2.1-2.5 and estimates (2.67)-(2.69), we conclude (2.64).

Differentiating (2.58) with respect to x , we have

$$\alpha(\rho^{\alpha-1}\rho_{xxxx})_t + \gamma\rho^{\gamma-1}\rho_{xxxx} = M_2(x, t), \quad (2.70)$$

where

$$M_2(x, t) = M_{1x}(x, t) - (\alpha - 1)(\rho^{\alpha-2}\rho_x\rho_{xxx})_t - \gamma(\gamma - 1)\rho_x\rho_{xxx}$$

and

$$M_{1x}(x, t) = M_{0xx}(x, t) - \gamma(\gamma - 1)(\rho_x\rho_{xx})_x - (\alpha - 1)(\rho^{\alpha-2}\rho_x\rho_{xx})_{tx}.$$

Using the interpolation inequality, and the embedding theorem, Lemmas 2.1-2.5, we can deduce that

$$\|M_2(t)\| \leq C_4(\|u_x(t)\|_{H^3} + \|\rho_x(t)\|_{H^3} + \|u_{tx}(t)\|_{H^2} + \|v_x(t)\|_{H^2}). \quad (2.71)$$

Inserting (2.45) into (2.71), and integrating (2.71) with respect to t over $[0, T]$, using Lemmas 2.1-2.5, we have

$$\int_0^t \|M_2(s)\|^2 ds \leq C_4 + C_2 \int_0^t \|\rho_{xxxx}(s)\|^2 ds, \quad \forall t \in [0, T]. \quad (2.72)$$

Multiplying (2.70) by $\rho^{\alpha-1}\rho_{xxxx}$ in $L^2[0, 1]$, we can get

$$\frac{\alpha}{2} \frac{d}{dt} \|\rho^{\alpha-1}\rho_{xxxx}\|^2 + \gamma \int_0^1 \rho^{\gamma+\alpha-2} \rho_{xxxx}^2 dx \leq C_2 \|M_2(t)\|^2, \quad (2.73)$$

which implies

$$\frac{d}{dt} \|\rho^{\alpha-1} \rho_{xxxx}\|^2 \leq C_2 \|M_2(t)\|^2. \quad (2.74)$$

Integrating (2.74) with respect to t over $[0, T]$, using (2.72), we conclude

$$\|\rho^{\alpha-1} \rho_{xxxx}\|^2 \leq C_4 + C_2 \int_0^t \|\rho_{xxxx}(s)\|^2 ds, \quad t \in [0, T], \quad (2.75)$$

which, by virtue of Gronwall's inequality, gives

$$\|\rho_{xxxx}(t)\|^2 \leq C_4, \quad t \in [0, T]. \quad (2.76)$$

Thus, we can obtain (2.65) by virtue of (2.75)-(2.76).

By (2.24)-(2.26), (2.64)-(2.65), (2.45)-(2.47), and Lemmas 2.1-2.5, we deduce that

$$\begin{aligned} & \|u_{xxxx}(t)\|^2 + \|v_{xxxx}(t)\|^2 + \|w_{xxxx}(t)\|^2 \\ & + \int_0^t (\|u_{txxx}\|^2 + \|v_{txxx}\|^2 + \|w_{txxx}\|^2)(s) ds \leq C_4, \quad t \in [0, T]. \end{aligned} \quad (2.77)$$

On the other hand, we differentiate (1.20)-(1.22) with respect to x three times, use Lemmas 2.1-2.5 and (2.64)-(2.65) to conclude, for any $t \in [0, T]$,

$$\|u_{xxxxx}(t)\| \leq C_4 (\|u_{txxx}(t)\| + \|u_x(t)\|_{H^3} + \|\rho_x(t)\|_{H^3} + \|v_x(t)\|_{H^3}), \quad (2.78)$$

$$\|v_{xxxxx}(t)\| \leq C_4 (\|v_{txxx}(t)\| + \|u_x(t)\|_{H^3} + \|\rho_x(t)\|_{H^3} + \|v_x(t)\|_{H^3}), \quad (2.79)$$

$$\|w_{xxxxx}(t)\| \leq C_4 (\|w_{txxx}(t)\| + \|u_x(t)\|_{H^3} + \|\rho_x(t)\|_{H^3} + \|w_x(t)\|_{H^3}). \quad (2.80)$$

Thus we conclude from (2.77)-(2.80), (2.64)-(2.65), and Lemmas 2.1-2.5 that

$$\int_0^t (\|u_{xxxxx}\|^2 + \|v_{xxxxx}\|^2 + \|w_{xxxxx}\|^2)(s) ds \leq C_4, \quad \forall t \in [0, T]$$

which, combined with (2.77), implies (2.66). The proof is complete. \square

Proof of Theorem 1.1. Applying Lemmas 2.1-2.6, we readily get estimate (1.27)-(1.30) and complete the proof of Theorem 1.1. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed to each part of this work equally.

Acknowledgements

This research was supported in part by NSFC (No. 11501199) and the Natural Science Foundation of Henan Province (No. 14B110037).

Received: 26 January 2016 Accepted: 28 March 2016 Published online: 12 April 2016

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